Absence of Cooper-type bound states in three- and few-electron systems

S.K. Adhikari[1,a,b] and T. Frederico[2]

[1] Instituto de Física Teórica, Universidade Estadual Paulista, 01.405-900 São Paulo, São Paulo, Brazil
[2] Departamento de Física, Instituto Tecnológico Aeronautica, Centro Técnico Aeroespacial,
12228-900 São José dos Campos, São Paulo, Brazil

Abstract. It is shown that the appearance of a fixed-point singularity in the kernel of the two-electron Cooper problem is responsible for the formation of the Cooper pair for an arbitrarily weak attractive interaction between two electrons. This singularity is absent in the problem of three and few superconducting electrons at zero temperature on the full Fermi sea. Consequently, such three- and few-electron systems on the full Fermi sea do not form Cooper-type bound states for an arbitrarily weak attractive pair interaction.

PACS. 74.20.Fg BCS theory and its development

1 Introduction

For an arbitrarily weak residual attractive interaction, at zero temperature two electrons over the full Fermi sea spontaneously form a bound Cooper pair [1]. These Cooper pairs lay the foundation of the microscopic Bardeen-Cooper-Schreiffer (BCS) theory of superconductivity [2]. Flux quantization and other experimental evidences support the formation of Cooper pairs and their role in the BCS theory. Appearance of a fixed-point singularity in the kernel of the momentum space Schrödinger equation is responsible for the formation of Cooper pairs for arbitrarily weak attractive interaction. Similar singularity also appears in a general two-fermion problem in vacuum in one and two dimensions and is responsible, in these cases, for the formation of two-fermion bound states for very weak attractive interactions. This singularity is absent in the two-fermion problem in vacuum in three dimensions, and hence in that case there is no two-fermion bound state for a very weak attractive interaction.

The possibility of few-fermion clustering at low temperature is of recent interest [4] in the BCS to Bose crossover problem [3] in the Fermi liquid model of not only a free electron gas but several other fermionic systems such as nuclear matter, neutron matter (neutron stars), and electron-hole systems. Although Cooper pairing is supposed to dominate the weak-coupling BCS limit, it is not clear that few- and multi-electron clustering are not allowed in different spin and angular momentum states for coupling simulating a crossover from the BCS to Bose

a John Simon Guggenheim Memorial Foundation Fellow.
b e-mail: `adhikari@axp.ift.unesp.br`

problem. This possibility is of concern as in the low-density region of nuclear matter it has been shown [4] that at low temperatures the dominant part of nuclear matter will form α particles which are more strongly bound than deuterons. Hence, spin-triplet (deuteron) Cooper pairing and subsequent condensation of deuterons has to compete with α-particle Cooper quartetting and subsequent condensation. Here, in view of the study of Cooper-type quartetting in nuclear matter, in this work we consider the possibility of few-electron clustering within the Fermi liquid model of electrons. The principal difference between the nuclear matter and the superconducting electron gas problem is that in the former the interaction is much stronger than the weak residual attraction in case of electrons. So far existence of few-electron Cooper-type bound states have not been experimentally confirmed. However, if few-electron Cooper-type bound states are allowed in a specific spin/angular momentum state (for example, in an exotic high-T_c superconductor), they should be correctly accomodated in any microscopic theory of superconductivity.

At zero temperature, a Cooper-type consideration of the three-electron problem on the full Fermi sea for a very weak attractive interaction shows that a fixed-point singularity of the type discussed above is absent in the kernel of the three-particle equation in momentum space. Hence there is no Cooper-type bound state for the three-electron system in any space dimension. Similar results should hold for clustering of n electrons with $n > 3$ on the full Fermi sea.

We consider the three-electron problem on the full Fermi sea with an arbitrarily weak two-electron interaction in S wave. In this case, for the state with total orbital

angular momentum of the three-electron system $L = 0$, obviously the effective interaction is repulsive because of the Pauli exclusion principle and there is no bound state. This is because a third electron, with spin up or down, can not approximate a singlet electron pair with one spin-up and one spin-down electron and form a three-electron bound state for $L = 0$, as the antisymmetrization of the three-electron wave function has to follow entirely from its spin part. However, for the three-electron state with total orbital angular momentum L odd, the antisymmetrization of the wave function follows partly from its spin part and partly from its orbital part. In this case, there could be an effective attraction between the three electrons in the state with $L = 1$, which could form a bound state. We write the Faddeev equation [5] for the three-electron problem on the full Fermi sea and find that the effective three-electron interaction is attractive for odd L states and repulsive for even L states for a very weak two-electron attractive interaction in S state. However, the Cooper-type singularity does not appear in the kernel of the Faddeev equation. Hence there are no Cooper-type three-electron bound states for very weak attractive interactions.

In Section 2 we present a discussion of the two-electron problem (a) in vacuum and (b) on the full Fermi sea for an arbitrarily weak attractive potential. We show that the presence of a fixed-point singularity in the kernel of the problem in momentum space is responsible for the formation of a bound state (a) in vacuum in one and two dimensions and (b) on the full Fermi sea in any dimension. In Section 3 we formulate the three-electron problem on the full Fermi sea for an arbitrarily weak attractive pair potential. We find that a fixed-point singularity does not appear in the kernel of the momentum-space Faddeev equation in this case and there is no three-electron bound state for an arbitrarily weak attractive pair potential.

2 The two-electron problem

For two electrons, each of mass m, in the center of mass frame the single (two) particle energy is given by $\epsilon_q = q^2$ $(2q^2)$, in units $\hbar = 2m = 1$, where q is the wave number. We consider a purely attractive weak residual S-wave short-range separable potential between electrons:

$$V(p, q) = -\lambda g(p)g(q). \tag{1}$$

Because of the presence of the lattice, the superconducting electrons experience a pairwise weak finite-range residual attraction which is modelled by the above potential. In the conventional BCS model the potential form factors $g(p)$ are set equal to unity and the range of the potential is introduced by the Debye cut off in momentum space [2]. The analysis and conclusion of this work are independent of this specific form of the potential employed. For Cooper pairing in zero (even) orbital angular momentum state(s), the allowed spin state of the two-electron system is $S = 0$ by Pauli exclusion principle. Hence this potential acts in the ^1S state: the S-wave spin-singlet state. The

two-electron problem, with this potential at energy E, is given by the following equation

$$f(q) = \int_c^\infty d^{\mathcal{D}}p \frac{V(q, p)f(p)}{E - 2p^2}, \tag{2}$$

where $c = 0$ in vacuum, and $c = k_F$, the Fermi momentum, for the Cooper problem. Here $G \equiv (E - 2p^2)^{-1}$ is the Green function, $f(q)$ the bound-state form factor, E the two-electron energy and $\mathcal{D}$ is the dimension of the space. For a two-electron bound state in vacuum, we have $E \leq 0$ and for the bound state over the full Fermi sea, the condition $E \leq 2k_F^2$ is to be satisfied.

In vacuum, for $\lambda \to 0$, the Green function develops a singularity at the lower limit as the binding energy $\alpha^2 (\equiv -E) \to 0$. In this limit, for the above separable potential, equation (2) reduces to

$$1 = \lambda C \int_0^\infty p^{\mathcal{D}-1} dp \frac{g^2(p)}{2p^2}, \tag{3}$$

where $C = 4\pi$ $(2\pi, 2)$ for $\mathcal{D} = 3$ $(2, 1)$. In three space dimensions, $(\mathcal{D} = 3)$, for usual well-behaved potential form factors $g(p)$, the integral in equation (3) is finite and it is impossible to satisfy condition (3) in the limit $\lambda \to 0$. Hence there are no bound states in vacuum for very weak potentials in three dimensions. However, in one and two dimensions, the integral in equation (3) is infinite and one can satisfy condition (3) in the limit $\lambda \to 0$ and one can have bound states in one and two dimensions for arbitrarily weak attractive potentials.

On the full Fermi sea, as $\lambda \to 0$, the Green function also develops a singularity at the lower limit as the Cooper pair binding energy $\alpha^2 (\equiv 2k_F^2 - E) \to 0$. In this limit, for the above separable potential, equation (2) reduces to

$$1 = \lambda C \int_{k_F}^\infty p^{\mathcal{D}-1} dp \frac{g^2(p)}{2(p^2 - k_F^2)} \cdot \tag{4}$$

Equation (4) represents the condition for Cooper instability. The integral in equation (4) is infinite in any space dimension for usual well-behaved form factors $g(p)$ and equation (4) can be satisfied in the limit $\lambda \to 0$ for any $k_F \neq 0$. For $k_F = 0$, the singularity in the Green function of equation (4) at the lower limit of the integral is cancelled by a zero in the phase space $p^{\mathcal{D}-1}dp$ for $\mathcal{D} = 3$. This singularity for $k_F = 0$ survives in the case of $\mathcal{D} = 1$ and 2 and is responsible for the two-electron bound state in vacuum for any arbitrarily weak attractive potential.

We base our discussion on equations of type (4) which led Cooper to his conclusion on pair formation. A more complete discussion could be based on a linearized version of the gap equation. The present simplification does not change anything in our qualitative discussion on the existence of Cooper-type states based of the appearance of singularities. The essential difference between the two approaches is in the use of the right phase space factors, which is not expected to change the general criteria for the appearance of singularities as considered here for two- and few-electron systems.

Consequently, one can have Cooper pairs in any space dimension for electrons interacting *via* a weak attractive interaction over the full Fermi sea. We have used a separable potential to reach the above conclusion. As the arguments are based on the existence of a singularity in the kernel, the conclusion should hold for any short-range potential. For potential (1), the two-electron t matrix for dynamics on the full Fermi sea has the following analytic form

$$t(p, q, E) = g(p)\tau(E)g(q), \qquad (5)$$

where

$$\tau(E) = \left[-\frac{1}{\lambda} - \int_{k_F}^{\infty} d^{\mathcal{D}}q g^2(q)(E - 2q^2)^{-1} \right]^{-1}. \qquad (6)$$

The use of separable potential (1) facilitates the solution of the three-electron problem on the full Fermi sea, which we take up in the next section.

3 The three-electron problem

The simplest of the three-electron problem on the full Fermi sea, that we consider here in some details, is the one where they interact *via* S-wave pair potential (1) in the ^1S state. This is the problem of three superconducting electrons on the top of the full Fermi sea at zero temperature and is in effect a many-body problem involving all the electrons and the lattice. However, the many-body nature of the problem introduces only minor changes over the three-electron problem in vacuum. As in the two-electron Cooper problem, one has a weak attractive interaction between the electrons in place of the Coulomb repulsion in vacuum with an appropriate truncation of the phase space in momentum space consistent with the Pauli principle. So effectively one has the problem of three electrons under the action of a weak attractive interaction in vacuum subject to the above-mentioned restrictions on the momentum-space phase space arising from the many-body nature of the problem.

The most likely assignment of the quantum number (LSJ) (total orbital angular momentum, total spin, and total angular momentum) for the bound state of the three electrons on the full Fermi sea in this case is $(L = 1, S = 1/2, J = 1/2$ and $3/2)$. The three-electron bound state can be visualized as the bound state of the third (spectator) particle with the spin-singlet bound state of the first two particles (1 and 2). For the spectator particle to be bound to the singlet pair, lowest value for its angular momentum state relative to the pair should be $\mathcal{L} = 1$. The value $\mathcal{L} = 0$ is not allowed by the Pauli exclusion principle. This is why the lowest probable value of L is 1. The only possible value for S is $1/2$, so that there are two degenerate J values $1/2$ and $3/2$. In the following we shall consider only this state. If, in addition, one allows a two-electron potential in the ^3P state, one can have a three-electron bound state for $(LSJ) = (1, 3/2, 1/2), (1, 3/2, 3/2), (0, 1/2, 1/2)$ *etc.* A complete analysis of these states, in the context of the three-neutron system, has been given by Mitra [6].

The t matrix of equation (5) acts as an effective potential in the three-electron Faddeev equation in three dimensions, which in this case can be written as [5,7]

$$F(\mathbf{q}, E) = 2\tau(E - 3q^2/2) \int d^3p K(\mathbf{q}, \mathbf{p}) F(\mathbf{p}, E), \qquad (7)$$

where

$$K(\mathbf{q}, \mathbf{p}) = \chi \frac{g(|\mathbf{p} + \mathbf{q}/2|)g(|\mathbf{q} + \mathbf{p}/2|)}{E - A}, \qquad (8)$$

with $A = p^2 + q^2 + (\mathbf{p} + \mathbf{q})^2$. Now E is the three-electron energy and for the bound state on the full Fermi sea $E \leq 3k_F^2$. Here, for $L = 1$ and $S = 1/2$, the spin coupling coefficient $\chi = -1/2$, and $p, q, |\mathbf{p} + \mathbf{q}| > k_F$. We should now stick to a specific total angular momentum state $\mathcal{L}$ (of the spectator particle) and take a partial-wave projection of equation (7) in $\mathcal{L}$ before solving it. The partial-wave projection of equation (7) is given by

$$F_{\mathcal{L}}(q, E) = 4\pi\tau(E - 3q^2/2) \int_{k_F}^{\infty} p^2 dp K_{\mathcal{L}}(q, p) F_{\mathcal{L}}(p, E), \qquad (9)$$

where

$$K_{\mathcal{L}}(q, p) = \int_{-1}^{1} dx P_{\mathcal{L}}(x) K(\mathbf{q}, \mathbf{p}) \Theta(A - 3k_F^2) \Theta(q^2 - k_F^2) \qquad (10)$$

is the partial-wave kernel for this problem. Here x is the cosine of the angle between $\mathbf{p}$ and $\mathbf{q}$, $P_{\mathcal{L}}(x)$ is the Legendre polynomial, and $\Theta(x) = 1$ for $x \geq 0$ and $= 0$, otherwise. Equation (10) with $k_F = 0$ is the usual kernel of the Faddeev equation. A direct calculation has revealed that, for an attractive potential, the kernel (10) is positive (negative) definite for $\mathcal{L}$ odd (even). Hence the three-body equation (9) is purely attractive for odd $\mathcal{L}$ and we shall consider $\mathcal{L} = 1$ below.

For a detailed study of this problem, one should consider the partial-wave equation (9). However, the full equation (7) reveals the essential interesting feature. The Green function of equation (8), unlike in the two-electron problem, does not have any fixed-point singularity for all q. In the arbitrarily weak attractive potential limit $\lambda \to 0$, the function τ of equations (5, 7) also tends to zero. Then one faces the question whether equation (7) permits a non-trivial solution in this limit, which could correspond to a weakly bound three-electron state. As the binding energy of the three-electron system is expected to tend to zero, the appropriate E in this equation is $E = 3k_F^2$. One can see from equation (7) that the energy denominator in this equation does not have a fixed-point singularity, as in the two-electron case. Hence the integral in equation (7) is finite and its right hand side is always zero in the weak-coupling limit as $\tau \to 0$. This means that the form factor $F(q, E)$ is identically zero in the weak-coupling limit and there is no three-electron bound state.

The situation does not change after the partial-wave projection. The kernel $K_{\mathcal{L}}(q, p)$ of equation (10) for $\mathcal{L} = 1$

develops weak and integrable moving logarithmic singularities at $2p^2 + 2q^2 \pm 2pq = 3k_F^2$, as opposed to fixed-point singularities in the two-electron problem. As these singularities are integrable, the integral on the right hand side of equation (9) is always finite, so that, in the limit of weak interaction, as $\tau \to 0$, this equation has the only trivial solution $F(q, E) = 0$. Hence there are no Cooper-type three-electron bound state in this limit. However, the kernel is attractive in this case and one can have a three-electron bound state as the strength of the potential is increased.

The present general discussion is based on the existence of a fixed-point singularity and not on some specific property of the system. Hence it should be applicable to other similar problems. For example, identical behavior is expected in one and two dimensions and other (LSJ) three-electron bound states on the full Fermi sea. So there are no Cooper type bound states in any space dimension for the three-electron system for an arbitrarily weak two-electron potential.

The present discussion can easily be extended to clustering of n electrons ($n \geq 4$) on the full Fermi sea. For $n = 4$, the problem should be treated by a four-body dynamical equation. The kernel of such equation will involve four- and three-electron Green functions, none of which could have a fixed-point singularity. Hence, by similar arguments, the four-electron Cooper-type bound states are not also allowed for an arbitrarily weak two-electron potential.

4 Summary

We have shown that the appearance of a fixed-point singularity in the kernel of the momentum-space Schrödinger equation is responsible for the existence of a bound state for an arbitrarily weak attractive potential in the zero-temperature limit. The formation of a Cooper pair in any space dimension and of a two-electron bound state in vacuum in one and two space dimensions under the action of a very weak attractive potential is due to the appearance of the above fixed-point singularity. Similar fixed-point singularities are not expected to appear in the n-electron ($n > 2$) problem and there are no Cooper-type bound states in these cases for arbitrarily weak attractive potentials in the zero-temperature limit. However, with stronger attractive potentials three and few-electon clusters can be formed on the full Fermi sea as has been discussed recently in the low-density limit of nuclear matter at low temperatures [4].

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